

## Energy Distribution in a Neutral Gas of Point Vortices

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An analytical formula for the energy distribution of a neutral gas of point vortices is obtained. Good agreement with the numerical results of Campbell and O'Neil is found.

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**KEY WORDS:** Point vortices; neutral vortex gas; energy distribution.

In ref. 1 the following problem was considered: Point vortices are thrown randomly and statistically independently into a rectangular cell and their positions are periodically duplicated over the plane. Point vortices generate some fluid flow. The kinetic energy of the flow per cell depends on the vortex positions, and, thus, is random. We are interested in the probability density function of energy. Knowing this function one can determine statistical characteristics of the vortex motion provided the latter is ergodic (see, for example, ref. 2). Campbell and O'Neil<sup>(1)</sup> computed the probability density function of the energy numerically by casting 160 point vortices randomly into a square box with periodic boundary conditions and then evaluating the energy of each configuration. In this note we present an analytical formula for the probability density function of the energy and compare it with the corresponding numerical results of Campbell and O'Neil.

Let an equal number of positive and negative point vortices be positioned in a cell  $C$ . Then the stream function of the flow obeys the equation

$$\Delta u = -\gamma \sum_{a=1}^N [\delta(x-r_a^+) - \delta(x-r_a^-)], \quad (1)$$

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where  $r_1^+, \dots, r_N^+$  and  $r_1^-, \dots, r_N^-$  are the positions of positive and negative vortices, respectively,  $\gamma$  is the vortex intensity, and  $\Delta$  is Laplace's operator. The function  $u$  and its derivatives are periodic.

This problem admits a variational formulation: one has to minimize the functional

$$I(u) = \frac{1}{2} (Au, u) - (l, u),$$

$$(Au, u) = \int_C (\nabla u)^2 d^2x, \quad (2)$$

$$(l, u) = \sum_{a=1}^N (l_0(r_a), u),$$

where  $r$  is a pair  $(r^+, r^-)$  and

$$(l_0(r^+, r^-), u) = \int_C \gamma [\delta(x - r^+) - \delta(x - r^-)] u(x) d^2x. \quad (3)$$

The minimum is sought on the subspace of  $H^1(C)$  which consists of periodic functions. Unfortunately, the variational problem is ill-posed as stated, for

$$\inf_u I(u) = -\infty,$$

because the kinetic energy of the flow generated by each individual point vortex is infinite. To have a sensible variational problem one has to regularize the functional by either smoothing the  $\delta$ -function or by adding higher derivatives with small parameter  $\epsilon$  to the functional

$$(A_\epsilon u, u) = \int_C [(\nabla u)^2 + \epsilon^2 (\nabla \nabla u)^2] d^2x. \quad (4)$$

Here  $(\nabla \nabla u)^2 \equiv \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}$ , Greek indices run values 1,2 and summation over repeated indices is implied. The constant  $\epsilon$  plays the role of a vortex core radius. The way of regularization does not matter since we look for results which do not depend on the core radius. We found the regularization (4) more convenient.

Note that the functional  $I_\epsilon(u)$  is invariant under a shift by a constant. To have a unique minimizing function we impose an additional constraint

$$\langle u \rangle = 0, \quad (5)$$

where  $\langle \cdot \rangle$  denotes the space averaging over the cell,  $\langle u \rangle = \frac{1}{|C|} \int_C u d^2x$ ,  $|C|$  being the area of the cell.

Let  $\mathbf{r}$  be the set  $(r_1^+, r_1^-, \dots, r_N^+, r_N^-)$ . Energy  $E$  is a function of  $\mathbf{r}$  and, thus, is a random variable. We are seeking the probability density function of the energy. Denote by  $\hat{E}_1$  the vortex self-energy, to be defined later, and by  $E' = (E - 2N\hat{E}_1)/N$  the interaction energy divided by the number of disjoint unlabeled neutral pairs,  $N$ . We claim that the probability density function of  $E'$ ,  $f_{N,\epsilon}(E')$ , which depends on  $N$  and  $\epsilon$ , has the limit when we let  $N \rightarrow \infty$  first and then  $\epsilon \rightarrow 0$ , and we determine this limit explicitly.

The general problem of finding the probability density function of the minimum values of stochastic functionals was solved in ref. 3. For the reader's convenience we recall the following result from ref. 3.

Consider a problem of minimizing the following quadratic functional:

$$I(u) = \frac{1}{2} (Au, u) - (l, u). \quad (6)$$

Here  $(Au, u)$  and  $(l, u)$  stand for the quadratic and linear parts of the functional  $I(u)$ ,  $u$  being an element of some Hilbert space. The linear functional is random, i.e. it depends on an "event"  $\mathbf{r}$ , where  $\mathbf{r}$  is an element of some set on which a probabilistic measure is given. We consider random linear functionals  $(l, u)$  of a special type:  $(l, u)$  is a sum of  $N$  values of a linear random functional  $l_0$

$$(l, u) = \sum_{a=1}^N (l_0(r_a), u),$$

where  $r_a$  are independently and identically distributed random variables. Denote by  $\hat{E}$  the quantity

$$\hat{E} = -\min_u I(u)/N. \quad (7)$$

It turns out that the probability density function of  $\hat{E}$ ,  $f_N(\hat{E})$ , does not depend on  $N$  for large  $N$ , and can be determined by an asymptotic formula<sup>(3)</sup>

$$f(\hat{E}) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\hat{E}z}}{\sqrt{\Phi(z)}} dz, \quad (8)$$

where

$$\Phi(z) = \prod_{k=1}^{\infty} \left( 1 + \frac{z}{\lambda_k} \right), \quad (9)$$

and  $\lambda_k$  are the eigenvalues  $\lambda_k$  of the following eigenvalue problem

$$A\varphi = \lambda B\varphi, \quad (10)$$

with the operator  $B$  defined by

$$(B\varphi, \varphi) = M((I_0, \varphi)^2).$$

$M$  stands for the mathematical expectation. In the infinite product (9) each eigenvalue is counted as many times as its multiplicity is. The integral (8) is taken over the imaginary axis. Note that by changing the variable,  $u \rightarrow u\sqrt{N}$ , formula (7) can be written in the form

$$\hat{E} = -\min_u \left[ \frac{1}{2} (Au, u) - \frac{1}{\sqrt{N}} \sum_{a=1}^N (I_0(r_a), u) \right].$$

This is the form which was used in ref. 3.

The function of a complex variable  $\hat{f}(z) = 1/\sqrt{\Phi(z)}$  is the characteristic function of the probability density function of the energy. In the case under consideration it can be found explicitly. Indeed, for the linear functional  $I_0$  from (3) one easily finds that

$$B\varphi = \frac{2\gamma^2}{L^2} \varphi.$$

Here we used the fact that computing of mathematical expectation is equivalent to averaging over the cell,  $M(\cdot) = \langle \cdot \rangle$ , and we took also into account the condition (5). The eigenvalue problem (10) now reads

$$-\Delta\varphi + \epsilon^2 \Delta^2\varphi = \lambda \frac{2\gamma^2}{L^2} \varphi. \quad (11)$$

It is convenient to make this equation dimensionless by introducing dimensionless coordinates and a dimensionless parameter  $\bar{\epsilon}$  according to

$$\bar{x} = \frac{x}{L}, \quad \bar{\epsilon} = \frac{\epsilon}{L}.$$

Then (11) takes the form

$$-\bar{\Delta}\varphi + \bar{\epsilon}^2 \bar{\Delta}^2\varphi = \lambda 2\gamma^2\varphi. \quad (12)$$

Here  $\bar{\Delta}$  is Laplace's operator in dimensionless  $\bar{x}$ -coordinates. The periodic solutions of (12) are given by

$$\varphi = ae^{i2\pi k \cdot \bar{x}}, \quad k \in \mathbb{Z}'_2. \quad (13)$$

We denote by  $\mathbb{Z}'_2$  the 2D square lattice with unit spacing, in which the point  $k = (0, 0)$  is excluded. Obviously, the function (13) for  $k = (0, 0)$  is not an eigenfunction due to (5). Substituting (13) into (12) we obtain

$$|k|^2 + \bar{\varepsilon}^2 4\pi^2 |k|^4 = \lambda_k \frac{\gamma^2}{2\pi^2}.$$

Each point of the lattice  $\mathbb{Z}'_2$  corresponds to one eigenfunction. We may set this correspondence, for example, in the following way. Consider a point  $(k_1, k_2)$  in the first quadrant,  $k_1 > 0, k_2 > 0$ . It is accompanied by associate points  $(k_1, -k_2), (-k_1, -k_2), (-k_1, k_2)$  in three other quadrants giving the same eigenvalue  $\lambda_k$ . We set

$$\begin{aligned} \varphi_1 &= a_1 \sin 2\pi(k_1 \bar{x}_1 + k_2 \bar{x}_2) & \text{for } k &= (k_1, k_2), \\ \varphi_2 &= a_2 \cos 2\pi(k_1 \bar{x}_1 - k_2 \bar{x}_2) & \text{for } k &= (k_1, -k_2), \\ \varphi_3 &= a_3 \cos 2\pi(k_1 \bar{x}_1 + k_2 \bar{x}_2) & \text{for } k &= (-k_1, -k_2), \\ \varphi_4 &= a_4 \sin 2\pi(k_1 \bar{x}_1 - k_2 \bar{x}_2) & \text{for } k &= (-k_1, k_2). \end{aligned}$$

For  $k_1 = 0, k_2 \neq 0$  we put  $\varphi_1 = a_1 \sin 2\pi k_2 \bar{x}_2$  for  $k_2 > 0$  and  $\varphi_2 = a_2 \cos 2\pi k_2 \bar{x}_2$  for  $k_2 < 0$ . Similarly, we put  $\varphi_1 = a_1 \sin 2\pi k_1 \bar{x}_1$  for  $k_1 > 0, k_2 = 0$  and  $\varphi_2 = a_2 \cos 2\pi k_1 \bar{x}_1$  for  $k_1 < 0, k_2 = 0$ . The coefficients  $a_i$  of the eigenfunctions are chosen from the normalization condition  $\langle \varphi_i^2 \rangle = 1$ . Introducing the following quantities

$$\varepsilon = 2\pi\bar{\varepsilon}, \quad e_0 = \frac{\gamma^2}{2\pi^2}, \quad \bar{\lambda}_k = \lambda_k e_0,$$

we get for  $\bar{\lambda}_k$  the simple equation

$$\bar{\lambda}_k = |k|^2 + \varepsilon^2 |k|^4.$$

The function  $\Phi(z)$  and the probability density of the energy  $f(\hat{E})$  depend on the parameter  $\varepsilon$ . To emphasize this in our notation we attach the label  $\varepsilon$  to each of these functions and write  $\Phi_\varepsilon(z)$  and  $f_\varepsilon(\hat{E})$ , respectively. It is convenient to write  $\Phi_\varepsilon(z)$  in the form

$$\Phi_\varepsilon(z) = e^{-2h_\varepsilon(z)}.$$

Then

$$f(\hat{E}) = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{\hat{E}z + h_\varepsilon(z)} dz. \quad (14)$$

For the function  $h_\varepsilon(z)$  we have

$$h_\varepsilon(z) = -\frac{1}{2} \ln \Phi_\varepsilon(z) = -\frac{1}{2} \sum_{k \in \mathbb{Z}'_2} \ln \left( 1 + \frac{z}{|k|^2 + \varepsilon^2 |k|^4} \right). \quad (15)$$

It is shown in the Appendix B that, if  $\varepsilon \rightarrow 0$ , the function  $h_\varepsilon(z)$  in (14) can be replaced by the function

$$h_\varepsilon(z) \sim \phi_0(z) - 2\hat{E}_1 z, \quad (16)$$

where

$$\phi_0(z) = -\frac{1}{2} \sum_{k \in \mathbb{Z}'_2} \left[ \ln \left( 1 + \frac{z}{|k|^2} \right) - \frac{z}{|k|^2} \right], \quad (17)$$

and  $\hat{E}_1$  corresponds to the self-energy of a single vortex (the notion of vortex self-energy is discussed in more details in Appendix A)

$$\hat{E}_1 = \sum_k \frac{1}{4\bar{\lambda}_k} = \sum_{k \in \mathbb{Z}'_2} \frac{1}{|k|^2 + \varepsilon^2 |k|^4}. \quad (18)$$

More precisely,

$$\int_{-\infty}^{i\infty} e^{\hat{E}z + h_\varepsilon(z)} dz - \int_{-\infty}^{i\infty} e^{\hat{E}z + \phi_0(z) - 2\hat{E}_1 z} dz \rightarrow 0 \quad (19)$$

as  $\varepsilon \rightarrow 0$  if the interaction energy,  $E' = \hat{E} - 2\hat{E}_1$ , is kept fixed. In other words, the probability density function of the interaction energy,  $f_\varepsilon(E')$ , has the limit as  $\varepsilon \rightarrow 0$ , and this limit is equal to

$$f(E') = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} e^{E'z + \phi_0(z)} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iE'y + \phi_0(iy)} dy. \quad (20)$$

Note that the self-energy  $\hat{E}_1$  tends to infinity as  $\varepsilon \rightarrow 0$ , and, in fact, we make “an infinite shift of energy”.

The computation of the integral (20) can be conducted in the following way. Since the series (17) converges slowly, we calculate  $\phi_0(z)$  by summing

up the terms with  $k$  inside a circle of radius  $\varrho$  ( $|k| < \varrho$ ) and approximating the rest of the series by a double integral

$$-\frac{1}{2} \sum_{|k| > \varrho} \left[ \ln \left( 1 + \frac{z}{|k|^2} \right) - \frac{z}{|k|^2} \right] \approx -\frac{1}{2} \int_{|k| > \varrho} \left[ \ln \left( 1 + \frac{z}{|k|^2} \right) - \frac{z}{|k|^2} \right] d^2k.$$

This integral can be found exactly. It is equal to

$$-\frac{\pi}{2} \int_{\varrho^2}^{\infty} \left[ \ln \left( 1 + \frac{z}{t} \right) - \frac{z}{t} \right] dt = -\frac{\pi}{2} \left[ (z + \varrho^2) \ln \frac{\varrho^2}{z + \varrho^2} + z \right].$$

Figures 1 and 2 shows the results of calculations of  $\text{Re } \phi_0(iy)$  and  $\text{Im } \phi_0(iy)$  for  $\varrho = 1, 2, 40$ . The convergence is fast, and for  $\varrho > 3$  we practically obtain the limit curve for  $\phi_0(iy)$ . The graph of  $f(E')$  calculated according to (20) is shown in Fig. 3. To obtain  $f(\hat{E})$  we have to shift this function to the right by  $2\hat{E}_1$ . Note that the mean value of  $E'$  is zero, which is simply the consequence of the periodic boundary condition. In Fig. 3 we show also the distribution of  $E'$  obtained numerically by Campbell and O'Neil.<sup>(1)</sup> The energy unit in ref. 1 is chosen as

$$e_{c-o} = \frac{\gamma^2}{2\pi},$$

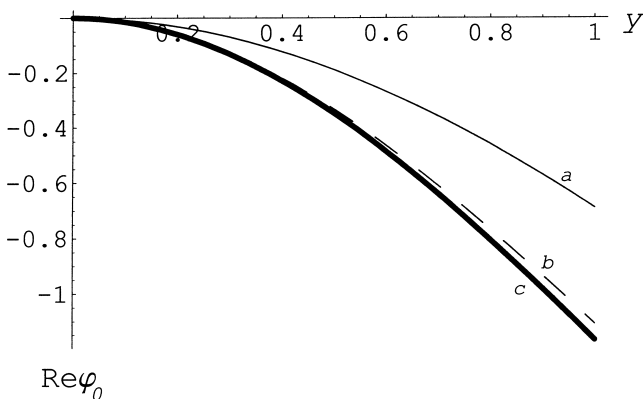


Fig. 1. Approximate calculation of function  $\text{Re } \phi_0(iy)$ : a)  $\varrho = 1$ , b)  $\varrho = 2$  (dashed line) c)  $\varrho = 40$  (bold line).

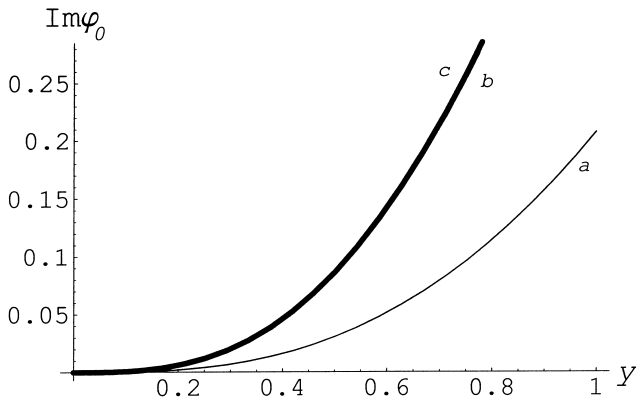


Fig. 2. Approximate calculation of function  $\text{Im } \phi_0(iy)$ : a)  $\varrho = 1$ , b)  $\varrho = 2$  (dashed line) c)  $\varrho = 40$  (bold line).

i.e. Campbell and O'Neil computed the energy  $\tilde{E}$  which differs from  $E'$  by the factor  $\pi$ :  $\tilde{E} = E'/\pi$ . Thus, if we denote the probability density function of  $\tilde{E}$  by  $\tilde{f}(\tilde{E})$ , then

$$f(E') = \frac{1}{\pi} \tilde{f}\left(\frac{E'}{\pi}\right).$$

The bold line in Fig. 3, which correspond to the formula (20), is in good agreement with the numerical results of Campbell and O'Neil.

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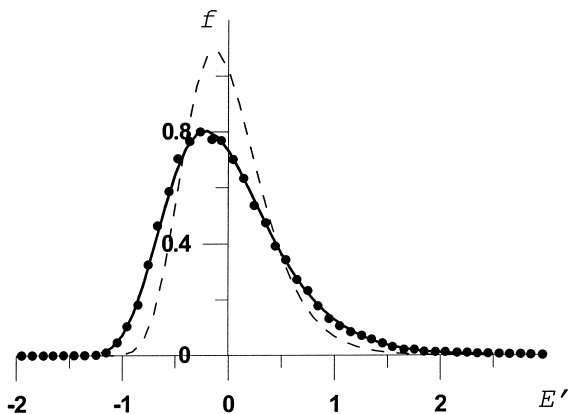


Fig. 3. Probability density function of interaction energy: a)  $\varrho = 1$  (dashed line), b) function (20) (bold line), c) numerical results of Campbell and O'Neil (points).



## APPENDIX A. VORTEX SELF-ENERGY

Consider a flow with an infinite periodic set of point vortices and a neutralizing constant background vorticity. The stream function of the flow  $G$  is the periodic solution of the equation

$$\Delta G - \epsilon^2 \Delta G = -\gamma \delta(x-r) + \frac{\gamma}{|C|} \quad (21)$$

Each cell contains one vortex. The neutralizing background vorticity  $\gamma/|C|$  is added to make the solution consistent with the periodic boundary conditions.

Let us expand  $G$  in the Fourier series with respect to the eigenfunctions  $\varphi_k$ ,

$$G(x-r) = \sum_{k \in \mathbb{Z}'_2} u_k(r) \varphi_k(x). \quad (22)$$

Plugging (22) in (21) we obtain

$$u_k(r) = \frac{1}{2\gamma\lambda_k} \varphi_k(r),$$

The energy of the flow per cell is given by the formula

$$\begin{aligned} \text{Energy} &= \frac{1}{2} \int_C [(\nabla G)^2 + \epsilon^2 (\nabla \nabla G)^2] d^2x \\ &= \frac{1}{2} \int_C G(-\Delta G + \epsilon^2 \Delta^2 G) d^2x \\ &= \frac{1}{2} \sum_{k \in \mathbb{Z}'_2} \frac{1}{2\gamma\lambda_k} \int_C \varphi_k(r) \varphi_k(x) \left( \gamma \delta(x-r) - \frac{\gamma}{|C|} \right) d^2x \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}'_2} \frac{1}{\lambda_k} \varphi_k^2(r). \end{aligned} \quad (23)$$

The energy does not depend on the vortex position in the cell due to the periodicity of the flow. Putting in (23)  $r=0$  and taking into account that  $\varphi_k(0) = 1$  we obtain

$$\text{Energy} = \frac{1}{4} \sum_{k \in \mathbb{Z}'_2} \frac{1}{\lambda_k} = \frac{e_0}{4} \sum_{k \in \mathbb{Z}'_2} \frac{1}{\bar{\lambda}_k}$$

This energy referred to the unit  $e_0$  coincides with the self-energy  $\hat{E}_1$  defined by (18). Note that  $\hat{E}_1 \rightarrow \infty$  as  $\ln 1/\varepsilon$ .

## APPENDIX B. PROOF OF (19)

We first note that the integral (14) converges absolutely for all  $\hat{E} > 0$ . Indeed, substituting  $z = iy$  in (14) we have

$$f_\varepsilon(\hat{E}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\hat{E}y + h_\varepsilon(iy)} dy. \quad (24)$$

Thus

$$|f_\varepsilon(\hat{E})| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\operatorname{Re} h_\varepsilon(iy)} dy. \quad (25)$$

Taking in (15) only terms with  $|k| = 1$  we obtain the inequality

$$\operatorname{Re} h_\varepsilon(iy) \leq -\ln \left( 1 + \frac{y^2}{(1 + \varepsilon^2)^2} \right),$$

which shows that the right-hand side of (24) is finite. Therefore, for  $\varepsilon \rightarrow 0$ , the integral (24) can be replaced by the integral over  $y \in (-1/\sqrt{\varepsilon}, 1/\sqrt{\varepsilon})$ . In this interval we decompose  $h_\varepsilon(z)$  into two parts

$$\begin{aligned} h_\varepsilon(z) &= \phi_\varepsilon(z) - 2\hat{E}_1 z, \\ \phi_\varepsilon(z) &= -\frac{1}{2} \sum_{k \in \mathbb{Z}'_2} \left[ \ln \left( 1 + \frac{z}{|k|^2 + \varepsilon^2 |k|^4} \right) - \frac{z}{|k|^2 + \varepsilon^2 |k|^4} \right], \end{aligned} \quad (26)$$

where  $\hat{E}_1$  was given by (18).

The function  $\phi_\varepsilon(iy)$  is regular within the interval  $(-1/\sqrt{\varepsilon}, 1/\sqrt{\varepsilon})$  and converges to

$$\phi_0(iy) = -\frac{1}{2} \sum_{k \in \mathbb{Z}'_2} \left[ \ln \left( 1 + \frac{iy}{|k|^2} \right) - \frac{iy}{|k|^2} \right] \quad (27)$$

as  $\varepsilon \rightarrow 0$ . To show this we use the identity

$$\chi_\varepsilon(z) = \phi_\varepsilon(z) - \phi_0(z) = -\frac{1}{2} z^2 \sum_{k \in \mathbb{Z}'_2} \int_0^{\varepsilon^2} \frac{d\tilde{\varepsilon}}{(|k|^2 + \tilde{\varepsilon} |k|^4 + z)(1 + \tilde{\varepsilon} |k|^2)^2}, \quad (28)$$

which is obtained by differentiating (26) with respect to  $\tilde{\varepsilon} = \varepsilon^2$  and then integrating over the interval  $\tilde{\varepsilon} \in (0, \varepsilon^2)$ . The left-hand side of (28), for all  $z = iy$ ,  $y \in (-1/\sqrt{\varepsilon}, 1/\sqrt{\varepsilon})$ , is bounded above by

$$\begin{aligned} |\chi_\varepsilon(iy)| &\leq \frac{1}{2} y^2 \sum_{k \in \mathbb{Z}'_2} \int_0^{\varepsilon^2} \frac{d\tilde{\varepsilon}}{\sqrt{(|k|^2 + \tilde{\varepsilon} |k|^4)^2 + y^2} (1 + \tilde{\varepsilon} |k|^2)^2} \\ &\leq \frac{1}{\sqrt{2}} y^2 \sum_{k \in \mathbb{Z}'_2} \int_0^{\varepsilon^2} \frac{d\tilde{\varepsilon}}{|k|^2 + \tilde{\varepsilon} |k|^4 + |y|} = \frac{1}{\sqrt{2}} y^2 \sum_{k \in \mathbb{Z}'_2} \frac{1}{|k|^4} \ln \left( 1 + \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \right). \end{aligned}$$

In order to give an upper bound for the series standing in the right-hand side of this inequality we decompose it into

$$\begin{aligned} \sum_{k \in \mathbb{Z}'_2} \frac{1}{|k|^4} \ln \left( 1 + \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \right) &= \sum_{|k| < \varrho} \frac{1}{|k|^4} \ln \left( 1 + \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \right) \\ &+ \sum_{|k| > \varrho} \frac{1}{|k|^4} \ln \left( 1 + \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \right) = A + B. \end{aligned}$$

Since  $\ln(1+x) < x$  we have for  $A$

$$A \leq \sum_{|k| < \varrho} \frac{1}{|k|^4} \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \leq \varepsilon^2 \sum_{|k| < \varrho} \frac{1}{|k|^2}.$$

We strengthen this inequality by replacing the sum by a double integral over  $k$ ,  $1 < |k| < \varrho$ , giving

$$A \leq \varepsilon^2 \int_{1 < |k| < \varrho} \frac{1}{|k|^2} d^2k \leq 2\pi\varepsilon^2 \ln \varrho. \quad (29)$$

$B$  is bounded above by a double integral

$$B \leq \int_{|k| > \varrho} \frac{1}{|k|^4} \ln \left( 1 + \frac{\varepsilon^2 |k|^4}{|k|^2 + |y|} \right) d^2k$$

which can be calculated exactly in the polar coordinates. As the result the following inequality holds for  $B$

$$B \leq \pi \frac{\ln(1 + \varepsilon^2 \varrho^2)}{\varrho^2} + \pi\varepsilon^2 \ln \frac{1 + \varepsilon^2 \varrho^2}{\varepsilon^2 \varrho^2}. \quad (30)$$

Choosing  $\varrho = 1/\varepsilon$  and combining (29) and (30), we see that

$$|\chi_\varepsilon(iy)| \leq \frac{1}{\sqrt{2}} y^2 (A + B) \leq \text{const} \frac{1}{\varepsilon} \varepsilon^2 \ln \frac{1}{\varepsilon} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . Thus, (19) holds true.

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